Chapter 6
Dimensional Analysis and Similarity

Fluid Mechanics involve many empirical forms that rely on data acquired by experimental means. The general equations solved on large computers, yield only fair approximation for turbulent flow problems, as an example, and the verifications with experimental results are needed.

In analyzing experimental results, it is essential to employ dimensionless parameters to save time, effort and costs. Using dimensionless parameters, figure 6-1 with three curves is reduced to one curve only, figure 6-2. Five dimensional parameters are reduced to two dimensionless variables.

In 1915 Buckingham * showed that the number of n dimensional variables in a specific problem can be reduced to n-m dimensionless variables, where m is the number of fundamental dimensions appear in the n variables. As a familiar example the pressure drop due to pipe friction Δ P is said to depend on fluid velocity V, fluid density ρ, pipe diameter D, pipe length L, surface roughness Є, and fluid viscosity μ. These seven dimensional variable (n=7) include three fundamental dimensions (M, L, T) m=3 can be reduced to 4 dimensionless variables as shown:

$$\Delta P = f(V, \rho, D, L, \varepsilon, \mu)$$

$$\frac{\Delta P}{\rho V^2} = F\left(\frac{L}{D}, \frac{\varepsilon}{D}, \frac{\rho V D}{\mu}\right)$$

Generally the theory does not predict the nature (linear, power, exponential etc) of the functions relations f nor F. It should be also noticed that these dimensionless variable are independent from each other.

<u>Step by Step Procedure to Find the dimensionless Π terms :</u>

1) **Arrange the n dimensional variables such that the dependent variable on the LHS and the independent variables on the RHS with their most important appearing first. These important variable should be important for the whole range of its value.**
2) **Identify the fundamental dimensions for each variable and evaluate m**
3) **Identify m variables to be repeated in all your dimensionless groups $\Pi_1 . \Pi_2\ \Pi_3 .. \Pi_{n-m}$**
4) **Express you Π_i term with 3 unknown exponents for the repeated variables with the non-repeated variable number i.**
5) **Equate powers of the fundamental dimensions e.g. M,L, and T on both sides to find the unknown exponents**

Example

The power consumed P by a fan is said to depend on the air density ρ, fan impeller diameter D speed of rotation N , the volume flow rate Q and the pressure increase across the fan Δp

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*****Buckingham, E ” Model Experiment and the forms of Empirical Equations.” ASME Trans., Vol 37, 1915, P.263

Show that

$$\frac{P}{\rho N^3 D^5} = F\left[\frac{Q}{ND^3}, \frac{\Delta P}{\rho N^2 D^2}\right]$$

Solution

We have 6 dimensional variables with the following dimensions
[P] = $ML^2 T^{-3}$ the square brackets are used to designate dimensions
[ρ] = ML^{-3}
[D] = L
[N] = T^{-1}
[Q] = L^3T^{-1}
P = f (ρ, D, N, Q, Δp) with ρ, D, and N are considered the most important independent variables.
$\Pi_1 = P \rho^a D^b N^c$ Substitute with the fundamental dimensions of each variable we get
$M^0 L^0 T^0 = M^{(1+a)} L^{(2-3a+b)} T^{(-3-c)}$
a =-1, c= -3 and b=-5
$\Pi_1 = P \rho^{-1} D^{-5} N^{-3} = P/(\rho^1 D^5 N^3)$ Similarly $\Pi_2 = Q/(ND^3)$ and $\Pi_3 = \Delta p/(N^2D^2)$

The followings are some dimensionless group of variables that are common in fluid mechanics

1) **Reynolds number Re It represents the ratio between inertia force and the viscous force; Re= ρVD/μ or VD/ ν**
2) **Froude number Fr. It represents the square root of the ratio between the inertia force and gravity force; Fr=V/√gl**
3) **Euler number Eu, it represents the ratio between pressure force and inertia force; Eu=P/(ρV^2)**
4) **Mach number M it represents the square root of ratio of inertia force and compressibility force M=V/C where C is the speed of sound in the fluid C=√(K/ρ).**
5) **Weber number We It represents the ratio of inertia force to surface tension force; We = $\rho V^2 l / \sigma$**

Dimensionless drag and lift coefficients are usually used as given below.

Drag and Lift

The sum of the forces that acts parallel to free stream direction is the lift L, while, the sum of the forces that act parallel to the free stream direction is the drag D.

The lift and drag coefficients are defined by;

$$C_d = \frac{D}{0.5\rho U^2 A}$$

$$C_l = \frac{L}{0.5\rho U^2 A}$$

where U is the upstream velocity, A is the body area projected on a plane normal to U and ρ is the fluid density.

Example:

The drag on a plane at a speed of 380 km/hr in a standard air is to be determined from tests on a 1:10 scale model placed in a pressurized wind tunnel. Model free stream velocity 190 m/s. Find he required pressure in the tunnel to achieve dynamic similarity. If the model drag force was 200 N. Evaluate the prototype drag.

Unsteady Flow in Pipes

When a valve is closed in a pipe a positive pressure wave is created upstream of the valve and travels downstream at the speed C. Depending on the closure time, the magnitude of the pressure may be much greater than the steady state pressure in the pipe. The density ρ is also increased by $\Delta\rho$. If the pipe is assumed to be rigid the wave will travel at the speed of sound and C is given by

$C = \sqrt{\frac{K}{\rho}}$ where K is the bulk modulus of elasticity

The above relation can be derived from the integral form of mass conservation as follows

$$0 = -\rho AV + \frac{d}{dt}[\rho(L - Ct)A + (\rho + \Delta\rho)ACt]$$

$$0 = -\rho AV - \rho AC + (\rho + \Delta\rho)AC$$

$$\frac{\Delta\rho}{\rho} = \frac{V}{C}$$

Apply the momentum conservation;

$$PA - (P + \Delta P)A = \frac{d}{dt}[(L - Ct)A\rho V] - \rho AV^2$$

$$\Delta P = \rho V^2 + \rho VC$$

The first term on the right hand side can be neglected with respect to the second term, and the pressure increase would be equal ρVC. If the elasticity of the pipe is considered with pipe material thickness t and Young modulus of elasticity E; Poisson ratio υ (ratio of lateral strain to longitudinal strain)

$$C = \sqrt{\frac{K/\rho}{1 + a(\frac{DK}{tE})}}$$

a depends on the constrains on the pipe due to the pipe fixation:

a = 1 for free pipe with expansion joint every where

a= $1.25 - \upsilon$ for pipe anchored upstream only.

a = $1 - \upsilon^2$ for pipe anchored both upstream and downstream.

The followings are values for E and υ for some common pipe materials.

Material	E	υ
Steel	220 G Pa	0.3
Ductile cast steel	176 G Pa	0.28
Copper	130 G Pa	0.36
Brass	110 G Pa	0.34

If the closure time T_{cl} is larger than the critical time $t_c = 2L/C$ then the closure would be gradual and when the wave return will not find the valve completely closed. A rule of thump for the magnitude of pressure increase $\Delta P = 2L\rho V/T_c$

As the closure time increases the magnitude of the pressure rise due to valve closure decreases.

Example:

A steel pipe of length 100 m and diameter 60 cm and thickness 4 mm has water average velocity 2 m/s. If the valve in the pipe is closed find the speed of pressure wave and the pressure increase if the closure time is 1) 0.2 s 2) 2 s Consider the above explained three methods of fixation (for water K=2.07 G Pa).

Solution:

The wave speed is given by $C = \sqrt{\dfrac{K/\rho}{1 + a(\dfrac{DK}{tE})}}$

		ΔP sudden
For free pipe	C = 926.52 m/s	1.853 M Pa
For pipe anchored upstream only	C= 940.38 m/s	1.881M Pa
For full restrained pipe	C= 951.93 m/s	1.904 M Pa

The critical time is $t_c = 2L/C = 0.21$ s

Closure in 0.2 s is considered sudden and ΔP = 1000(V)(C)

While closure in 2 s is considered gradual and $\Delta P = 1000(V)(2L/T_{cl}) = 0.2$ M Pa irrespective of fixation method.